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AUTHOR(S):

Takayama, Masahiro

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Regularity of solutions to non-uniformly characteristic boundary value problems for symmetric systems

高山 正宏 (Masahiro Takayama)

慶應義塾大学 理工学部

(Faculty of Science and Technology, Keio University)

1. Introduction

The purpose of this paper is to the study of the regularity of solutions to boundary value problems for first order symmetric systems with non-uniformly characteristic boundary. Let Ω be a bounded open subset of \mathbf{R}^n ($n \geq 2$) with smooth boundary $\partial\Omega$. We consider first order symmetric systems of the form

$$Lu = \sum_{j=1}^n A_j(x) \partial_j u + B(x)u, \quad A_j(x), B(x) \in C^\infty(\overline{\Omega}), \quad A_j^*(x) = A_j(x)$$

where $u = (u_1, \dots, u_N)$ and $\partial_j = \partial/\partial x_j$. We study the following boundary value problem;

$$(BVP) \quad \begin{cases} (L + \lambda)u = f & \text{in } \Omega \\ u(x) \in M(x) & \text{at } \partial\Omega \end{cases}$$

where $M(x)$ ($x \in \partial\Omega$) is a linear subspace of \mathbf{C}^N which is maximal non-negative in the sense that

$$\begin{aligned} \langle A_b(x)v, v \rangle &\geq 0 \quad \text{for all } v \in M(x), \\ \dim M(x) &= \#\{\text{non-negative eigenvalues of } A_b(x) \text{ counting multiplicity}\}. \end{aligned}$$

The boundary matrix is given by

$$A_b(x) = \sum_{j=1}^n \nu_j A_j(x) \quad (x \in \partial\Omega)$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the unit outward normal to Ω .

A general theory for the boundary value problems (BVP) has been developed by many authors. The case of non-characteristic boundary (that is, the boundary matrix $A_b(x)$ is non-singular everywhere on $\partial\Omega$) has been studied by Friedrichs [2], Lax-Phillips [5], Tartakoff [16], Rauch-Massey III [12] and so on. The case of uniformly characteristic boundary (that is, $A_b(x)$ is singular but has constant rank on $\partial\Omega$) has been treated by Lax-Phillips [5], Rauch [11], Yanagisawa-Matsumura [18], Ohno-Shizuta-Yanagisawa [10] and so on.

Our main concern is the case of non-uniformly characteristic boundary (that is, $A_b(x)$ changes the rank on $\partial\Omega$). The existence of weak solutions to (BVP) is classical. The regularity of solutions to (BVP) has been studied by Nishitani-Takayama [6], [7] and Secchi

[14], [15]. To explain the details, assume that there is an embedded $n - 2$ dimensional submanifold γ of $\partial\Omega$ such that the rank of $A_b(x)$ is constant in each component of $\partial\Omega \setminus \gamma$. The case when $A_b(x)$ is positive definite on one side of $\partial\Omega \setminus \gamma$ and negative definite on the other side is studied in [6], [14].

In this paper, we consider the same problem when the rank of $A_b(x)$ changes simply crossing γ . We study the following two cases:

- (I) $A_b(x)$ is non-singular in $\partial\Omega \setminus \gamma$ and definite on one side of $\partial\Omega \setminus \gamma$.
- (II) The rank of $A_b(x)$ is constant in $\partial\Omega \setminus \gamma$ and $A_b(x)$ vanishes on γ .

In general, even for smooth f , solutions u to (BVP) is not necessarily regular because singularities of u may occur on the characteristic curves passing through points of tangency on the boundary (see [6, Example 2.1], [14, Example 4]). Hence, to get regularity results, we impose further conditions (see Sections 2 and 3).

The case (I) is also studied in [7]. The result, expressed in terms of weighted conormal Sobolev spaces, implies the normal regularity of weak solutions only at a part of the boundary. In this paper we prove the normal regularity of weak solutions at the boundary outside γ under the same assumptions as in [6]. In the case (II), we can also obtain the normal regularity of weak solutions outside γ if $A_b(x)$ is non-singular on $\partial\Omega \setminus \gamma$. But we need another observation different from that of the case (I).

The plan of this paper is as follows: We state our main results in Sections 2 and 3 with several examples. From Section 5 through Section 7 we first study the case (I) and prove Theorems 2.1, 2.2 and 2.3. From Section 8 to Section 10 we next study the case (II) and prove Theorems 3.1 and 3.2.

In what follows, we denote by $r(x)$ a smooth function with $dr(x) \neq 0$ on $\partial\Omega$ so that $\Omega = \{r(x) > 0\}$ and by $h(x)$ a smooth function such that $\gamma = \partial\Omega \cap \{h(x) = 0\}$ where $dh(x)$ and $\nu(x)$ are linearly independent on γ .

2. Assumptions and Main Results (I)

We first consider the case (I). We make our assumptions precise. Let us set

$$O^+(O^-) = \{x \in \partial\Omega; A_b(x) \text{ is positive (negative) definite}\}$$

and denote by γ^\pm the smooth boundaries of O^\pm in $\partial\Omega$. In the case (I) we may assume that $\gamma = \gamma^+ \cup \gamma^-$ and that $A_b(x)$ is non-singular outside γ . We assume also that $\text{Ker} A_b(x)$ is a C^∞ vector bundle over γ . Let $\{v_1(x), \dots, v_p(x)\}$ be a smooth basis for $\text{Ker} A_b(x)$ on γ (we may assume that $v_i(x)$ is defined in a neighborhood of γ). Since the matrix $(\langle A_b(x)v_i(x), v_j(x) \rangle)_{i,j=1,\dots,p}$ vanishes on γ , so one can factor out $h(x)$ so that

$$(\langle A_b(x)v_i(x), v_j(x) \rangle)_{i,j=1,\dots,p} = h(x)A_\gamma(x) \text{ in a neighborhood of } \gamma$$

where the right-hand side defines $A_\gamma(x)$. We next define $\tilde{A}_h(x)$ by

$$\tilde{A}_h(x) = (\langle A_h(x)v_i(x), v_j(x) \rangle)_{i,j=1,\dots,p}$$

where $A_h(x) = \sum_{j=1}^n (\partial_j h)(x) A_j(x)$. In the case (I) our assumption is stated as:

$$(2.1) \quad A_\gamma(x) \text{ and } \tilde{A}_h(x) \text{ have the same definiteness on } \gamma.$$

Under this assumption we get an existence and a regularity result on (BVP).

Take an $h_{\pm}(x) \in C^{\infty}(\overline{\Omega})$ such that $O^{\pm} = \partial\Omega \cap \{h_{\pm}(x) > 0\}$ where $dh_{\pm}(x)$ and $\nu(x)$ are linearly independent on γ^{\pm} . Let us set

$$m(x) = \{r(x)^2 + h(x)^2\}^{1/2}, \quad m_{\pm}(x) = \{r(x)^2 + h_{\pm}(x)^2\}^{1/2}, \\ \phi_{\pm}(x) = \{r(x)^2 + h_{\pm}(x)^2 + h_{\pm}(x)^4\}^{1/2} - h_{\pm}(x).$$

Note that $\phi_{\pm}(x) > 0$ if $x \in \overline{\Omega} \setminus \gamma^{\pm}$ and that $\phi_{\pm}(x) = 0$ if $x \in \gamma^{\pm}$. We now introduce the following spaces: For $q \in \mathbf{Z}_+$ and $\sigma, \tau \in \mathbf{R}$ we define

$$X_{(\sigma, \tau)}^q(\Omega) = \bigcap_{j=0}^q \phi_+^{\sigma+q-j} \phi_-^{\tau+q-j} H^j(\Omega), \\ X_{(\sigma, \tau)}^q(\Omega; \partial\Omega) = \bigcap_{j=0}^q \phi_+^{\sigma+q-j} \phi_-^{\tau+q-j} H^j(\Omega; \partial\Omega)$$

where $H^j(\Omega)$ and $H^j(\Omega; \partial\Omega)$ denote the usual Sobolev space of order j and the conormal Sobolev space of order j with respect to $\partial\Omega$ respectively (these conormal Sobolev spaces are studied in Section 4 below).

Theorem 2.1. *For $q \in \mathbf{Z}_+$ there is an $s(q) > 0$ such that for $\sigma, \tau > s(q)$ we can choose a $\Lambda(q, \sigma, \tau) \in \mathbf{R}$ having the following properties: If $f \in X_{(-\sigma, \tau)}^q(\Omega; \partial\Omega) \cap \phi_- L^2(\Omega)$ and $\text{Re } \lambda > \Lambda(q, \sigma, \tau)$ then there exists a weak solution $u \in X_{(-\sigma, \tau)}^q(\Omega; \partial\Omega) \cap \phi_- L^2(\Omega)$ to (BVP) which satisfies*

$$\|u\|_{X_{(-\sigma, \tau)}^q(\Omega; \partial\Omega)} + \|\phi_-^{-1} u\|_{L^2(\Omega)} \leq C \{\|f\|_{X_{(-\sigma, \tau)}^q(\Omega; \partial\Omega)} + \|\phi_-^{-1} f\|_{L^2(\Omega)}\}$$

where $C = C(q, \sigma, \tau, \lambda) > 0$ is independent of f and u .

Further we can get a rough estimate of the asymptotic behavior of solutions near γ .

Theorem 2.2. *For $q \in \mathbf{Z}_+$ there is an $s(q) > 0$ such that for $\sigma, \tau > s(q)$ one can take a $\Lambda(q, \sigma, \tau) \in \mathbf{R}$ with the following properties: If $f \in X_{(-\sigma, \tau)}^q(\Omega) \cap \phi_- L^2(\Omega)$ and $\text{Re } \lambda > \Lambda(q, \sigma, \tau)$ and if $u \in m_- L^2(\Omega)$ is a weak solution to (BVP) then it follows that $u \in m^{-q} \phi_+^{-\sigma} \phi_-^{\tau} H^q(\Omega)$.*

Since $m(x) > 0$ and $\phi_{\pm}(x) > 0$ if $x \in \overline{\Omega} \setminus \gamma^{\pm}$, this theorem implies the normal regularity at $\partial\Omega$ of weak solutions outside γ .

We remark that solutions u to (BVP) need not belong to $H^q(\Omega)$ even for $f \in C_0^{\infty}(\Omega)$.

Example 2.1 Let us set $\Omega = \{x_1^2 + x_2^2 < 1\}$ and consider

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \partial_1 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \partial_2 + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad \left(A_b(x) = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \right).$$

In this case, γ consists of four points $(\pm 1, 0), (0, \pm 1)$. Note that the condition (2.1) is fulfilled. A maximal positive boundary space $M(x)$ is

$$M(x) = \begin{cases} \mathbf{C}^2 & \text{if } x_1 > 0, x_2 > 0 \\ \{0\} \times \mathbf{C} & \text{if } x_1 < 0, x_2 > 0 \\ \{0\} & \text{if } x_1 < 0, x_2 < 0 \\ \mathbf{C} \times \{0\} & \text{if } x_1 > 0, x_2 < 0. \end{cases}$$

Now let us choose a $\chi \in C_0^\infty(\mathbf{R})$ so that

$$\chi(s) = 1 \quad \text{if } |s| < \epsilon, \quad \chi(s) = 0 \quad \text{if } |s| > 2\epsilon$$

where $\epsilon > 0$ is small enough and define the functions $g(x) = (g_1(x), g_2(x))$ and $v(x) = (v_1(x), v_2(x))$ in Ω as

$$\begin{aligned} g_1(x) &= \chi(x_1)\chi(x_2), \quad g_2(x) = 0, \\ v_1(x) &= \int_{-\infty}^{x_1} \chi(s)ds \chi(x_2), \quad v_2(x) = \int_{-\infty}^{x_1} \chi(s)ds \int_{-\sqrt{1-x_1^2}}^{x_2} \chi(s)ds. \end{aligned}$$

Take a $\lambda \in \mathbf{R}$ and set $f(x) = e^{-\lambda(x_1+x_2)}g(x)$ and $u(x) = e^{-\lambda(x_1+x_2)}v(x)$. Then it is easy to see that u is a weak solution to (BVP).

We now work near $(1, 0)$. If $|x_2| < \epsilon$ and $x_1 > \sqrt{1-\epsilon^2}$ then $v_2(x) = c_0(x_2 + \sqrt{1-x_1^2})$ where $c_0 = \int_{-\infty}^{\infty} \chi(s)ds$, and hence we have $u \notin H^2(\Omega)$ in spite of $f \in C_0^\infty(\Omega)$. At the same time, it is easily checked that

$$u \in m^{-q}H^{q+1}(\Omega), \quad u \notin m^{-q}H^{q+2}(\Omega)$$

for $q \in \mathbf{Z}_+$. Thus this fact suggests Theorem 2.3 is sharp in a sense.

3. Assumptions and Main Results (II)

We next consider the case (II). We make our assumptions precise. Since $A_b(x)$ vanishes on γ , so one can factor out $h(x)$ so that

$$(3.1) \quad A_b(x) = h(x)A_\gamma(x) \quad \text{in a neighborhood of } \gamma$$

where the right-hand side defines $A_\gamma(x)$. Our first assumption is:

$$(3.2) \quad \text{the rank of } A_\gamma(x) \text{ is constant in a neighborhood of } \gamma.$$

Moreover, to get regularity results, we impose another condition as follows:

$$(3.3) \quad A_h(x) \text{ vanishes on } \gamma$$

where $A_h(x) = \sum_{j=1}^n (\partial_j h)(x) A_j(x)$.

As for the boundary condition we can write

$$M(x) = \begin{cases} M_+(x) & \text{on } \Gamma_+ := \partial\Omega \cap \{h(x) > 0\} \\ M_-(x) & \text{on } \Gamma_- := \partial\Omega \cap \{h(x) < 0\}. \end{cases}$$

We assume that $M_\pm(x)$ is smooth in Γ_\pm up to the boundary and

$$(3.4) \quad \dim[M_+(x) \cap M_-(x)] \text{ is constant on } \gamma.$$

Under the assumptions (3.2), (3.3) and (3.4) we get the following regularity results.

Let us set

$$m(x) = \{r(x)^2 + h(x)^2\}^{1/2}.$$

For $q \in \mathbf{Z}_+$ we denote by $H^q(\Omega; \gamma)$ (resp. $H^q(\Omega; \partial\Omega, \gamma)$) the conormal Sobolev space of order q with respect to γ (resp. $\partial\Omega$ and γ) (these spaces are defined and studied in Section 3).

Theorem 3.1. For $q \in \mathbf{Z}_+$ and $\sigma \geq 0$ there is a $\Lambda(q, \sigma) \in \mathbf{R}$ having the following properties: If $f \in m^\sigma H^q(\Omega; \partial\Omega, \gamma)$ and $\operatorname{Re} \lambda > \Lambda(q, \sigma)$ and if $u \in L^2(\Omega)$ is a weak solution to (BVP) then it follows that $u \in m^\sigma H^q(\Omega; \partial\Omega, \gamma)$ and

$$\|m^{-\sigma} u\|_{H^q(\Omega; \partial\Omega, \gamma)} \leq C \|m^{-\sigma} f\|_{H^q(\Omega; \partial\Omega, \gamma)}$$

where $C = C(q, \sigma, \lambda) > 0$ is independent of f and u .

Furthermore, if $A_\gamma(x)$ is non-singular on γ , we obtain

Theorem 3.2. For $q \in \mathbf{Z}_+$ and $\sigma \geq 0$ there is a $\Lambda(q, \sigma) \in \mathbf{R}$ having the following properties: If $f \in m^\sigma H^q(\Omega; \gamma)$ and $\operatorname{Re} \lambda > \Lambda(q, \sigma)$ and if $u \in L^2(\Omega)$ is a weak solution to (BVP) then it follows that $u \in m^\sigma H^q(\Omega; \gamma)$ and

$$\|m^{-\sigma} u\|_{H^q(\Omega; \gamma)} \leq C \|m^{-\sigma} f\|_{H^q(\Omega; \gamma)}$$

where $C = C(q, \sigma, \lambda) > 0$ is independent of f and u .

To get regularity results we could not replace $H^q(\Omega; \partial\Omega, \gamma)$ and $H^q(\Omega; \gamma)$ by $H^q(\Omega; \partial\Omega)$ or $H^q(\Omega)$ in Theorems 3.1 and 3.2.

Example 3.1 Let us consider $L = x_2 \partial_1 - x_1 \partial_2$ in $\Omega = \mathbf{R}_+^2$ with $h(x) = x_2$. Since $A_b(x) = -x_2$, $A_h(x) = -x_1$ and $\gamma = (0, 0)$ so the conditions (3.2) and (3.3) are fulfilled. The maximal positive boundary space $M(x)$ is

$$M(x) = \begin{cases} \{0\} & \text{if } x_1 = 0, x_2 > 0 \\ \mathbf{C} & \text{if } x_1 = 0, x_2 < 0. \end{cases}$$

Now let us take a $\lambda > 0$ and choose a $\chi \in C_0^\infty(\mathbf{R}^2)$ so that $\chi \equiv 1$ near the origin. We define $v(x)$ in \mathbf{R}_+^2 as $v(x) = \lambda^{-1}(1 - e^{\lambda(\tan^{-1}(x_2/x_1) - \pi/2)})$ and set $u = \chi v$ and $f = \chi + vL\chi$. Then u is a weak solution to (BVP). On the other hand we have $u \notin H^1(\mathbf{R}_+^2; \partial\mathbf{R}_+^2)$ in spite of $f \in H^\infty(\mathbf{R}_+^2)$.

We give another example of vector field showing an analogous result above of which flow, though, is completely different from that of Example 3.1.

Example 3.2 Let us consider $L = x_2 \partial_1 + x_1 \partial_2$ in $\Omega = \mathbf{R}_+^2$ with $h(x) = x_2$. Similarly, since $A_b(x) = -x_2$, $A_h(x) = x_1$ and $\gamma = (0, 0)$ so the conditions (3.2) and (3.3) are fulfilled. The maximal positive boundary space $M(x)$ is the same one as in Example 3.1 above. Let us take a $\lambda > 0$ and choose a $\chi \in C_0^\infty(\mathbf{R}^2)$ so that $\chi \equiv 1$ near the origin. We define $v(x)$ in \mathbf{R}_+^2 as

$$v(x) = \begin{cases} \lambda^{-1}(1 - (\frac{x_2 - x_1}{x_2 + x_1})^{\lambda/2}) & \text{if } 0 < x_1 < x_2 \\ \lambda^{-1} & \text{otherwise} \end{cases}$$

and set $u = \chi v$ and $f = \chi + vL\chi$. Then u is a weak solution to (BVP). On the other hand we have $u \notin H^1(\mathbf{R}_+^2; \partial\mathbf{R}_+^2)$ in spite of $f \in H^q(\mathbf{R}_+^2)$ (taking $\lambda > 0$ large enough we may assume $q \geq 1$).

4. Preliminaries

For the proof of main results, we shall localize the problem. Let $\{U_i\}$, $\{\chi_i\}$ and $\{\psi_i\}$ be the covering of Ω , the coordinate systems and the partition of unity, respectively. Suppose that $u \in L^2(\Omega)$ is a weak solution to (BVP). Then $u_i = \psi_i u$ is also a weak solution to (BVP). Therefore it suffices to show main results with u_i instead of u . The proof of the case $U_i \cap \gamma = \emptyset$ is much easier than that of the case of $U_i \cap \gamma \neq \emptyset$. Thus the interesting patches are at γ . In what follows, we write simply U, u for U_i, u_i and consider the case of $U \cap \gamma \neq \emptyset$. Performing a change of independent variables we are led to the case that

$$\Omega = \mathbf{R}_+^n = \{x \in \mathbf{R}^n; x_1 > 0\}, \quad \gamma = \{(0, 0, x''); x'' \in \mathbf{R}^{n-2}\}$$

$$r(x) = x_1, \quad h(x) = x_2, \quad U = \{|x| < 1\}, \quad \text{supp } u \subset \mathbf{R}_+^n \cap U$$

where $x = (x_1, x') = (x_1, x_2, x'') = (x_1, x_2, x_3, \dots, x_n)$.

By α, α' we denote multi-indices, that is, $\alpha \in \mathbf{Z}_+^n, \alpha' \in \mathbf{Z}_+^{n+2}$. With

$$Z = (Z_1, \dots, Z_n) = (x_1 \partial_1, \partial_2, \dots, \partial_n),$$

$$Z' = (Z'_1, \dots, Z'_{n+2}) = (x_1 \partial_1, x_2 \partial_2, \partial_3, \dots, \partial_n, x_1 \partial_2, x_2 \partial_1)$$

we set

$$Z^\alpha = Z_1^{\alpha_1} \dots Z_n^{\alpha_n}, \quad Z'^{\alpha'} = Z'_1{}^{\alpha'_1} \dots Z'_{n+2}{}^{\alpha'_{n+2}}.$$

We now introduce the conormal Sobolev spaces. For $q \in \mathbf{Z}_+$ we set

$$\begin{aligned} H^q(\mathbf{R}_+^n; \partial \mathbf{R}_+^n) &= \{w \in L^2(\mathbf{R}_+^n); Z^\alpha w \in L^2(\mathbf{R}_+^n), |\alpha| \leq q\}, \\ H^q(\mathbf{R}_+^n; \gamma) &= \{w \in L^2(\mathbf{R}_+^n); Z'^{\alpha'} w \in L^2(\mathbf{R}_+^n), |\alpha'| \leq q\}, \\ H^q(\mathbf{R}_+^n; \partial \mathbf{R}_+^n, \gamma) &= \{w \in L^2(\mathbf{R}_+^n); Z'^{\alpha'} w \in L^2(\mathbf{R}_+^n), |\alpha'| \leq q, \alpha'_{n+2} = 0\}. \end{aligned}$$

These allow us to norm $H^q(\mathbf{R}_+^n; \partial \mathbf{R}_+^n)$, $H^q(\mathbf{R}_+^n; \gamma)$ and $H^q(\mathbf{R}_+^n; \partial \mathbf{R}_+^n, \gamma)$ as follows:

$$\begin{aligned} \|w\|_{H^q(\mathbf{R}_+^n; \partial \mathbf{R}_+^n)}^2 &= \sum_{|\alpha| \leq q} \|Z^\alpha w\|_{L^2(\mathbf{R}_+^n)}^2, \\ \|w\|_{H^q(\mathbf{R}_+^n; \gamma)}^2 &= \sum_{|\alpha'| \leq q} \|Z'^{\alpha'} w\|_{L^2(\mathbf{R}_+^n)}^2, \\ \|w\|_{H^q(\mathbf{R}_+^n; \partial \mathbf{R}_+^n, \gamma)}^2 &= \sum_{\substack{|\alpha'| \leq q \\ \alpha'_{n+2} = 0}} \|Z'^{\alpha'} w\|_{L^2(\mathbf{R}_+^n)}^2. \end{aligned}$$

As for the operator L , we may assume that

$$Lu = \sum_{j=1}^n A_j(x) \partial_j u + B(x)u, \quad A_j(x), B(x) \in \mathcal{B}^\infty(\overline{\mathbf{R}_+^n}), \quad A_j^*(x) = A_j(x)$$

(note that $A_h(x) = A_2(x)$). Since $A_b(x') = -A_1(0, x')$ for $(0, x') \in \partial \mathbf{R}_+^n$ we can write

$$(4.1) \quad Lu = -A_b(x) \partial_1 u + \tilde{A}(x) Z_1 u + \sum_{j=2}^n A_j(x) Z_j u + B(x)u, \quad \tilde{A}(x) \in \mathcal{B}^\infty(\overline{\mathbf{R}_+^n}).$$

5. Proof of Main Results (I)

We start with the proof of main results (I). We first give the proof of Theorem 2.1 admitting the following proposition:

Proposition 5.1. *For $q \in \mathbf{Z}_+$, $q \geq 1$ there are $c_0 = c_0(q) > 0$ and $s(q) > 0$ such that for $\sigma, \tau > s(q)$ we can take a $\Lambda(q, \sigma, \tau) \in \mathbf{R}$ verifying the following properties: If*

$$f \in m^{-2}X_{(-\sigma+1, \tau+1)}^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n) \cap L^2(\mathbf{R}_+^n)$$

and $\operatorname{Re} \lambda > \Lambda(q, \sigma, \tau)$ and if

$$u \in m^{-2}X_{(-\sigma+1, \tau+1)}^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n) \cap L^2(\mathbf{R}_+^n)$$

with $\operatorname{supp} u \subset \{x_1 > 0, |x| < 1\}$ and $\operatorname{supp} u \cap \gamma^- = \emptyset$ is a weak solution to (BVP), then it follows that

$$(5.1) \quad \phi_+^\sigma \phi_-^{-\tau} m^2 u \in H^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n)$$

and the estimate

$$(5.2) \quad \begin{aligned} & (\min(\sigma, \tau) - s(q)) \|\phi_+^\sigma \phi_-^{-\tau} m^2 u\|_{\mathbf{R}_{+,q-1,tan,\delta}^n}^2 \\ & \leq c_0 \|\phi_+^\sigma \phi_-^{-\tau} m^2 (L + \lambda) u\|_{\mathbf{R}_{+,q-1,tan,\delta}^n}^2 \\ & + C_1 \{ \|m^2 (L + \lambda) u\|_{X_{(-\sigma+1, \tau+1)}^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n)}^2 + \|(L + \lambda) u\|_{L^2(\mathbf{R}_+^n)}^2 \\ & + \|m^2 u\|_{X_{(-\sigma+1, \tau+1)}^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n)}^2 + \|u\|_{L^2(\mathbf{R}_+^n)}^2 \} \end{aligned}$$

holds for $0 < \delta \leq 1$ where $C_1 > 0$ depends only on q, σ, τ, λ and $\operatorname{supp} u$. Here the norm $\|\cdot\|_{\mathbf{R}_{+,q-1,tan,\delta}^n}$ is as in [7, Section 3].

Proof of Theorem 2.1. Proposition 5.1 implies that

Proposition 5.2. *For $q \in \mathbf{Z}_+$ there is an $s(q) > 0$ such that for $\sigma, \tau > s(q)$ we can take a $\Lambda(q, \sigma, \tau) \in \mathbf{R}$ having the following properties: If $f \in m^{-2}X_{(-\sigma, \tau)}^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n) \cap L^2(\mathbf{R}_+^n)$ and $\operatorname{Re} \lambda > \Lambda(q, \sigma, \tau)$ and if $u \in L^2(\mathbf{R}_+^n)$ with $\operatorname{supp} u \subset \{x_1 > 0, |x| < 1\}$ and $\operatorname{supp} u \cap \gamma^- = \emptyset$ is a weak solution to (BVP) then it follows that $u \in m^{-2}X_{(-\sigma, \tau)}^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n)$.*

Using Proposition 5.2 and repeating the same arguments as in [7, Section 11], we can complete the proof of Theorem 2.1. \square

Theorem 2.2 follows easily from [7, Proposition 2.2] and Theorem 2.1. Theorem 2.3 is an immediate corollary to Theorem 2.2 and Proposition 5.3 below.

Proposition 5.3. *Let $u \in X_{(-\sigma, \tau)}^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n)$ and $(L + \lambda)u \in X_{(-\sigma, \tau)}^q(\mathbf{R}_+^n)$ for some $q \in \mathbf{Z}_+$ and $\sigma, \tau \in \mathbf{R}$. Then it follows that $u \in m^{-q}X_{(-\sigma, \tau)}^q(\mathbf{R}_+^n)$ and*

$$\|m^q u\|_{X_{(-\sigma, \tau)}^q(\mathbf{R}_+^n)} \leq C \{ \|u\|_{X_{(-\sigma, \tau)}^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n)} + \|(L + \lambda)u\|_{X_{(-\sigma, \tau)}^q(\mathbf{R}_+^n)} \}$$

where $C = C(q, \sigma, \tau, \lambda) > 0$ is independent of u .

The proof of this proposition is given in [8, Proposition 4.4].

6. Estimate of Commutators

In what follows, we shall show Proposition 5.1. We may assume that $h_{\pm} = \pm x_2$ and that $\text{supp} u \subset U_{1-\zeta_0,0}^{\pm}$ with $x = (x_1, x') = (x_1, x_2, x'')$ and $\zeta_0 > 0$ small enough where

$$U_{R,\eta}^+ = \{x; |x| < R, x_1 \geq 0\}, \quad U_{R,\eta}^- = \{x; |x| < R, x_1 \geq 0, x_1^2 + x_2^2 > \eta\}$$

with $0 < R \leq 1$ and $0 \leq \eta \leq 1$ (for convenience sake we use the notation $U_{R,\eta}^+$, which is actually independent of η). If $U \cap \gamma^+ \neq \emptyset$, then performing a change of dependent variables we may assume that

$$A_b(x') = \begin{pmatrix} x_2 I_p & 0 \\ 0 & I_{N-p} \end{pmatrix}$$

for $(0, x') \in \partial \mathbf{R}_+^n = \partial \Omega$ (see [7, Section 6]). If $U \cap \gamma^- \neq \emptyset$, the boundary value problem can be also transformed into a similar one.

We first examine (5.1) of Proposition 5.1. Since

$$|\partial^{\alpha}(\phi_+^{\sigma} \phi_-^{\tau})| \leq C \phi_+^{\sigma-|\alpha|} \phi_-^{\tau-|\alpha|} \quad \text{on } \{|x| < 1\}$$

with some $C = C(\sigma, \tau, \alpha) > 0$, the assertion (5.1) is easily checked (see [7, Section 6]). We turn to the estimate (5.2). For this purpose, we introduce the conormal mollifier. Let us take a $\chi \in C_0^{\infty}(\mathbf{R}^n)$ so that $\text{supp} \chi \subset \{y; |y| < \zeta_0, y_2 > 0\}$ and set $\chi_{\epsilon}(y) = \epsilon^{-n} \chi(y/\epsilon)$ for $0 < \epsilon \leq 1$. We define $J_{\epsilon} : L^2(\mathbf{R}_+^n) \rightarrow L^2(\mathbf{R}_+^n)$ by

$$(6.1) \quad J_{\epsilon} w(x) = \int_{\mathbf{R}^n} w(x_1 e^{-y_1}, x' - y') e^{-y_1/2} \chi_{\epsilon}(y) dy$$

It is easily checked that $[Z_j, J_{\epsilon}] = 0$ and $J_{\epsilon} w \in H^{\infty}(\mathbf{R}_+^n; \partial \mathbf{R}_+^n) = \bigcap_{j=0}^{\infty} H^j(\mathbf{R}_+^n; \partial \mathbf{R}_+^n)$.

The following estimate is the key to proving Proposition 5.1 (see [9, Section 7]).

Proposition 6.1. *There are $c, s_0 > 0$ such that for $\sigma, \tau > s_0$ we can take a $\Lambda(\sigma, \tau) \in \mathbf{R}$ with the following properties: If $\text{Re} \lambda > \Lambda(\sigma, \tau)$ and if $u \in L^2(\mathbf{R}_+^n)$ with $\text{supp} u \subset U_{1-\zeta_0,0}^{\pm}$ is a weak solution to (BVP) then there is a $\epsilon_0 > 0$ which depends only on $\text{supp} u$ such that the estimate*

$$(\min(\sigma, \tau) - s_0) \|\phi_+^{\sigma} \phi_-^{\tau} J_{\epsilon} m^2 u\|_{L^2(\mathbf{R}_+^n)}^2 \leq c \|m \phi_+^{\sigma} \phi_-^{\tau} (L + \lambda) J_{\epsilon} m^2 u\|_{L^2(\mathbf{R}_+^n)}^2$$

holds for all $0 < \epsilon \leq \epsilon_0$.

To show Proposition 5.1, we must control terms such as $x_i(L + \lambda)J_{\epsilon} m^2 u$ ($i = 1, 2$). Let us recall that the maps $\# : L^2(\mathbf{R}_+^n) \rightarrow L^2(\mathbf{R}^n)$ and $\natural : L^{\infty}(\mathbf{R}_+^n) \rightarrow L^{\infty}(\mathbf{R}^n)$ defined by $w^{\#}(x) = w(e^{x_1}, x') e^{x_1/2}$ and $a^{\natural}(x) = a(e^{x_1}, x')$ which are norm preserving bijections. It is easy to see that

$$\begin{aligned} (aw)^{\#} &= a^{\natural} w^{\#}, \quad (J_{\epsilon} w)^{\#} = \chi_{\epsilon} * w^{\#}, \quad \partial_j(a^{\natural}) = (Z_j a)^{\natural} \quad (j = 1, \dots, n), \\ \partial_j(w^{\#}) &= \begin{cases} (Z_1 w)^{\#} + w^{\#}/2 & (j = 1) \\ (Z_j w)^{\#} & (j = 2, \dots, n). \end{cases} \end{aligned}$$

We now study $(x_i(L + \lambda)J_{\epsilon} m^2 u)^{\#}$.

Lemma 6.2. Let $u \in D_{1-\zeta_0,0}^\pm(\mathbf{R}_+^n)$. Then for all $0 < \epsilon \leq 1$ it follows that

$$\text{supp}(u^\#(x-y)\chi_\epsilon(y)) \subset \{(x,y); x_1 < 0, |x'| < 1, |y| < \zeta_0\}$$

where

$$D_{R,\eta}^\pm(\mathbf{R}_+^n) = \{u \in L^2(\mathbf{R}_+^n); Lu \in L^2(\mathbf{R}_+^n), \text{supp} u \subset U_{R,\eta}^\pm\}$$

for $0 < R \leq 1$ and $0 \leq \eta \leq 1$.

Let us take a $\psi \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ such that $\psi(x,y) \equiv 1$ if $x_1 \leq 0, |x'| \leq 1$ and $|y| \leq \zeta_0$ and $\text{supp} \psi \subset \{(x,y); x_1 < 1, |x'| < 2, |y| < 2\zeta_0\}$. Lemma 6.2 implies that we may cut off $u^\#(x-y)\chi_\epsilon(y)$ by ψ if necessary. We denote by $a(x,y)$, which differs from line to line, an element in $\mathcal{B}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and by $\|\cdot\|$ the norm in $L^2(\mathbf{R}_+^n)$ or in $L^2(\mathbf{R}^n)$ if there is no confusion.

Proposition 6.3. For $u \in D_{1-\zeta_0,0}^\pm(\mathbf{R}_+^n)$ we can write $(x_i(L+\lambda)J_\epsilon m^2 u)^\#$, $i = 1, 2$ as a sum of the following terms:

$$(6.2) \quad \int a(x,y)(m^2(L+\lambda)u)^\#(x-y)\chi_\epsilon(y)dy,$$

$$(6.3) \quad \int a(x,y)(m^2 u)^\#(x-y)\chi_\epsilon(y)dy,$$

$$(6.4) \quad \int a(x,y)(x_i u)^\#(x-y)y^\alpha \chi_\epsilon(y)dy,$$

$$(6.5) \quad \lambda \int a(x,y)(m^2 u)^\#(x-y)y^\alpha \chi_\epsilon(y)dy,$$

$$(6.6) \quad \epsilon^{-1} \int a(x,y)(m^2 u)^\#(x-y)y^\alpha (\partial_j \chi)_\epsilon(y)dy,$$

$$(6.7) \quad \int a(x,y)(x_i(L+\lambda)u)^\#(x-y)y^\beta \chi_\epsilon(y)dy,$$

$$(6.8) \quad \int a(x,y)u^\#(x-y)y^\beta \chi_\epsilon(y)dy,$$

$$(6.9) \quad \lambda \int a(x,y)(x_i u)^\#(x-y)y^\beta \chi_\epsilon(y)dy,$$

$$(6.10) \quad \epsilon^{-1} \int a(x,y)(x_i u)^\#(x-y)y^\beta (\partial_j \chi)_\epsilon(y)dy,$$

$$(6.11) \quad (x_i x_{i'})^\natural(x) \int a(x,y)u^\#(x-y)\chi_\epsilon(y)dy$$

where $i, i' = 1, 2, j = 1, \dots, n, |\alpha| = 1, |\beta| = 2$.

Proof. We can write

$$(6.12) \quad \begin{aligned} (x_i(L+\lambda)J_\epsilon m^2 u)^\# &= ([x_i(L+\lambda), J_\epsilon]m^2 u)^\# \\ &\quad + (J_\epsilon[x_i(L+\lambda), m^2]u)^\# + (J_\epsilon x_i m^2(L+\lambda)u)^\#. \end{aligned}$$

Clearly the third term on the right-hand side of (6.12) can be written as (6.2). Hence we first study the second term on the right-hand of (6.12). Since

$$[x_i(L+\lambda), m^2] = 2x_i x_1 A_1 + 2x_i x_2 A_2,$$

it suffices to examine $I_{i,i'} = (J_\epsilon x_i x_{i'} Au)^\#$ with $A(x) \in \mathcal{B}^\infty(\mathbf{R}^n)$. Note that we can write

$$\begin{aligned} I_{1,1} &= e^{2x_1} \int e^{-2y_1} (Au)^\#(x-y) \chi_\epsilon(y) dy, \\ I_{1,2} &= e^{x_1} x_2 \int e^{-y_1} (Au)^\#(x-y) \chi_\epsilon(y) dy - \int (x_1 Au)^\#(x-y) y_2 \chi_\epsilon(y) dy, \\ I_{2,2} &= x_2^2 \int (Au)^\#(x-y) \chi_\epsilon(y) dy - 2 \int (x_2 Au)^\#(x-y) y_2 \chi_\epsilon(y) dy \\ &\quad - \int (Au)^\#(x-y) y_2^2 \chi_\epsilon(y) dy. \end{aligned}$$

From $(Au)^\# = A^\natural u^\#$ the second term on the right-hand side of (6.12) can be written as a sum of (6.4), (6.8) and (6.11).

We turn to the first term on the right-hand side of (6.12). From (4.1) it suffices to study the following terms:

$$([A, J_\epsilon] m^2 u)^\#, \quad ([AZ_j, J_\epsilon] m^2 u)^\#, \quad ([\lambda A, J_\epsilon] m^2 u)^\#, \quad ([x_2 A_b \partial_1, J_\epsilon] m^2 u)^\#.$$

As argued in [7, Proposition 8.2], we see that these terms can be written as a sum of (6.3), (6.5) and (6.6) except the last term which can be written as a sum of the following terms:

$$(6.13) \quad \int a(x, y) (A_b \partial_1 m^2 u)^\#(x-y) y^\alpha \chi_\epsilon(y) dy,$$

$$(6.14) \quad \int a(x, y) (\partial_1 m^2 u)^\#(x-y) y^\beta \chi_\epsilon(y) dy.$$

Recalling (4.1) and noticing $\partial_{x_j} u^\#(x-y) = -\partial_{y_j} u^\#(x-y)$ we can write (6.13) as a sum of (6.3), (6.5), (6.6) and the following term:

$$\int a(x, y) ((L + \lambda) m^2 u)^\#(x-y) y^\alpha \chi_\epsilon(y) dy$$

which again can be written as a sum of (6.2) and (6.4).

It only remains to examine (6.14). Since $\partial_1 m^2 u = 2x_1 u + x_1 Z_1 u + x_2^2 \partial_1 u$, we can write (6.14) as a sum of (6.4) and the following terms:

$$(6.15) \quad \int a(x, y) (x_1)^\natural (x-y) (Z_1 u)^\#(x-y) y^\beta \chi(y) dy,$$

$$(6.16) \quad \int a(x, y) (x_2^2 \partial_1 u)^\#(x-y) y^\beta \chi(y) dy.$$

It is clear that (6.15) can be written as a sum of (6.4), (6.8) and (6.10). Moreover using $x_2^2 \partial_1 = x_2 \tilde{A}(x') A_b(x') \partial_1$ with

$$\tilde{A}(x') = \begin{pmatrix} I_p & 0 \\ 0 & x_2 I_{N-p} \end{pmatrix},$$

we can write (6.16) as a sum of (6.4), (6.7), (6.8), (6.9) and (6.10). \square

7. Proof of Proposition 5.1

We complete the proof of Proposition 5.1. Let $q \in \mathbf{Z}_+$, $q \geq 1$ and suppose that

$$u \in m^{-2} X_{(-\sigma+1, \tau+1)}^{q-1}(\mathbf{R}_+^n; \partial \mathbf{R}_+^n) \cap D_{1-\zeta_0, \eta}^\pm(\mathbf{R}_+^n)$$

is a weak solution to (BVP) with $f \in m^{-2}X_{(-\sigma+1, \tau+1)}^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n)$. We may assume that $\sigma, \tau \geq q+2$. Moreover we assume that χ in (6.1) satisfies

$$\begin{aligned}\hat{\chi}(\xi) &= O(|\xi|^{q+1}) \quad (\xi \rightarrow 0), \\ \hat{\chi}(t\xi) &= 0 \quad \text{for all } t \in \mathbf{R} \text{ implies } \xi = 0.\end{aligned}$$

The following three lemmas will be frequently used in the following.

Lemma 7.1. *There is a $C = C(\chi, q) > 0$ such that for all $0 < \epsilon_0 \leq 1$, $0 < \delta \leq 1$ and $w \in H^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n)$ it follows that*

$$\|w\|_{\mathbf{R}_+^n, q-1, \tan, \delta}^2 \leq C \left\{ \int_0^{\epsilon_0} \|J_\epsilon w\|_{L^2(\mathbf{R}_+^n)}^2 \epsilon^{-2q} (1 + \delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon + (1 + \epsilon_0^{-2}) \|w\|_{\mathbf{R}_+^n, q-1, \tan}^2 \right\}$$

where the norms $\|\cdot\|_{\mathbf{R}_+^n, q-1, \tan, \delta}$ and $\|\cdot\|_{\mathbf{R}_+^n, q-1, \tan}$ are as in [7, Section 3].

Lemma 7.2. *Let $a(x, y) \in \mathcal{B}^\infty(\mathbf{R}^n \times \mathbf{R}^n)$. Then for $\alpha \in \mathbf{Z}_+^n$ there is a $C = C(\chi, q, a, \alpha) > 0$ with the following properties: If $w \in H^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n)$ and if we set*

$$W_\epsilon(x) = \int_{\mathbf{R}^n} a(x, y) w^\#(x - y) y^\alpha \chi_\epsilon(y) dy$$

then for all $0 < \epsilon_0 \leq 1$ and $0 < \delta \leq 1$ we have

$$\int_0^{\epsilon_0} \|W_\epsilon\|^2 \epsilon^{-2q} (1 + \delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon \leq \begin{cases} C \|w\|_{\mathbf{R}_+^n, q-1, \tan, \delta}^2 & \text{if } |\alpha| = 0 \\ C \|w\|_{H^{q-|\alpha|}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n)}^2 & \text{if } 1 \leq |\alpha| \leq q \\ C \|w\|_{L^2(\mathbf{R}_+^n)}^2 & \text{if } |\alpha| \geq q+1. \end{cases}$$

Lemma 7.3. *For $0 < \eta \leq 1$ There are $\epsilon_0 = \epsilon_0(\eta) > 0$ and $C = C(\eta) > 0$ such that if $w \in D_{1-\zeta_0, \eta}^\pm(\mathbf{R}_+^n)$ then it follows that*

$$(\phi_+)^{\mathfrak{h}}(x - y + \theta y) \leq C, \quad (\phi_-^{-1})^{\mathfrak{h}}(x - y + \theta y) \leq C$$

for all $(x, y) \in \text{supp}(u^\#(x - y)\chi_\epsilon(y))$, $0 < \epsilon \leq \epsilon_0$ and $0 \leq \theta \leq 1$.

Lemmas 7.1 and 7.2 follow from [4, Theorem 2.4.1] and [7, Lemma 9.3]. Lemma 7.3 is easily checked.

Let $\epsilon_0 = \epsilon_0(\eta) > 0$ as in Lemma 7.3. Throughout this section, we denote by c_0 constants which depend only on q and by C_1 constants which depend on q, σ, τ, λ and η .

Proof of Proposition 5.1. It follows from Proposition 6.1 that

$$(\min(\sigma, \tau) - s_0) \|\phi_+^\sigma \phi_-^{-\tau} J_\epsilon m^2 u\|^2 \leq c \|m \phi_+^\sigma \phi_-^{-\tau} (L + \lambda) J_\epsilon m^2 u\|^2.$$

Now using Taylor's formula we have

$$\begin{aligned}(\phi_+^\sigma \phi_-^{-\tau} J_\epsilon m^2 u)^\#(x) &= \int (\phi_+^\sigma \phi_-^{-\tau})^{\mathfrak{h}}(x) (m^2 u)^\#(x - y) \chi_\epsilon(y) dy \\ &= \sum_{|\beta| \leq q} (\beta!)^{-1} \int ((Z^\beta \phi_+^\sigma \phi_-^{-\tau}) m^2 u)^\#(x - y) y^\beta \chi_\epsilon(y) dy \\ &\quad + \sum_{|\beta|=q+1} (\beta!)^{-1} (q+1) \int \Phi_\beta(x, y) (m^2 u)^\#(x - y) y^\beta \chi_\epsilon(y) dy \\ &= \sum_{|\beta| \leq q} U_\beta(x) + \sum_{|\beta|=q+1} U_\beta(x)\end{aligned}$$

$$\Phi_\beta(x, y) = \int_0^1 (1 - \theta)^q (Z^\beta \phi_+^\sigma \phi_-^\tau)^\sharp(x - y + \theta y) d\theta.$$

If $|\beta| = 0$ we can write

$$U_\beta(x) = \int (\phi_+^\sigma \phi_-^\tau m^2 u)^\sharp(x - y) \chi_\epsilon(y) dy = (J_\epsilon \phi_+^\sigma \phi_-^\tau m^2 u)^\sharp(x).$$

This implies that

$$(7.1) \quad \begin{aligned} & (\min(\sigma, \tau) - s_0) \left\{ \int_0^{\epsilon_0} \|J_\epsilon \phi_+^\sigma \phi_-^\tau m^2 u\|^2 \epsilon^{-2q} (1 + \delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon \right. \\ & \quad \left. + (1 + \epsilon_0^{-2}) \|\phi_+^\sigma \phi_-^\tau m^2 u\|_{\mathbf{R}_+^n, q-1, \tan}^2 \right\} \\ & \leq c \int_0^{\epsilon_0} \|m \phi_+^\sigma \phi_-^\tau (L + \lambda) J_\epsilon m^2 u\|^2 \epsilon^{-2q} (1 + \delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon \\ & \quad + C_1 \left\{ \sum_{1 \leq |\beta| \leq q+1} \int_0^{\epsilon_0} \|U_\beta\|^2 \epsilon^{-2q} (1 + \delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon + \|\phi_+^\sigma \phi_-^\tau m^2 u\|_{\mathbf{R}_+^n, q-1, \tan}^2 \right\}. \end{aligned}$$

Recalling (5.1) and using Lemma 7.1 we can prove that the left-hand side of (7.1) is bounded from below by

$$c_0^{-1} (\min(\sigma, \tau) - s_0) \|\phi_+^\sigma \phi_-^\tau m^2 u\|_{\mathbf{R}_+^n, q-1, \tan, \delta}^2.$$

We turn to the right-hand side of (7.1). We first consider the terms which contain U_β . If $1 \leq |\beta| \leq q$ then it follows from Lemma 7.2 that

$$\begin{aligned} \int_0^{\epsilon_0} \|U_\beta\|^2 \epsilon^{-2q} (1 + \delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon & \leq c_0 \|(Z^\beta \phi_+^\sigma \phi_-^\tau) m^2 u\|_{H^{q-|\beta|}(\mathbf{R}_+^n; \partial \mathbf{R}_+^n)}^2 \\ & \leq C_1 \|m^2 u\|_{X_{(-\sigma+1, \tau+1)}^{q-1}(\mathbf{R}_+^n; \partial \mathbf{R}_+^n)}^2. \end{aligned}$$

If $|\beta| = q + 1$ then noticing $\sigma, \tau \geq q + 2$ and using Lemmas 7.2 and 7.3 we can obtain

$$\int_0^{\epsilon_0} \|U_\beta\|^2 \epsilon^{-2q} (1 + \delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon \leq C'_1 \|u\|.$$

Furthermore since $\|\phi_+^\sigma \phi_-^\tau m^2 u\|_{\mathbf{R}_+^n, q-1, \tan}^2 \leq C_1 \|m^2 u\|_{X_{(-\sigma+1, \tau+1)}^{q-1}(\mathbf{R}_+^n; \partial \mathbf{R}_+^n)}^2$ we have

$$\begin{aligned} & (\min(\sigma, \tau) - s_0) \|\phi_+^\sigma \phi_-^\tau m^2 u\|_{\mathbf{R}_+^n, q-1, \tan, \delta}^2 \\ & \leq c_0 \int_0^{\epsilon_0} \|m \phi_+^\sigma \phi_-^\tau (L + \lambda) J_\epsilon m^2 u\|^2 \epsilon^{-2q} (1 + \delta^2/\epsilon^2)^{-1} d\epsilon/\epsilon \\ & \quad + C_1 \{ \|m^2 u\|_{X_{(-\sigma+1, \tau+1)}^{q-1}(\mathbf{R}_+^n; \partial \mathbf{R}_+^n)}^2 + \|u\|^2 \}. \end{aligned}$$

We next consider the first term on the right-hand side. Note that

$$\begin{aligned} \|m \phi_+^\sigma \phi_-^\tau (L + \lambda) J_\epsilon m^2 u\|^2 & \leq c \sum_{i=1}^2 \|x_i \phi_+^\sigma \phi_-^\tau (L + \lambda) J_\epsilon m^2 u\|^2 \\ & = c \sum_{i=1}^2 \|(\phi_+^\sigma \phi_-^\tau)^\sharp(x_i (L + \lambda) J_\epsilon m^2 u)^\sharp\|^2. \end{aligned}$$

Using Proposition 6.3 we estimate the right-hand side. In particular, we study the terms of type (6.11). It follows from $|x_i x_{i'}| \leq c m^2$ that $|(x_i x_{i'})^h| \leq c(m^2)^h$, and hence we have

$$\begin{aligned} & \|(\phi_+^\sigma \phi_-^\tau)^h(\cdot)(x_i x_{i'})^h(\cdot) \int a(\cdot, y) u^\#(\cdot - y) \chi_\epsilon(y) dy\|^2 \\ & \leq c \|(\phi_+^\sigma \phi_-^\tau m^2)^h(\cdot) \int a(\cdot, y) u^\#(\cdot - y) \chi_\epsilon(y) dy\|^2. \end{aligned}$$

Applying Taylor's formula to $(\phi_+^\sigma \phi_-^\tau m^2)^h(x)$ and repeating the same arguments as above we can get the desired estimate (5.2). Thus we complete the proof of Proposition 5.1, and hence we obtain main results (I). \square

8. Proof of Main Results (II)

Next we give the proof of main results (II). Theorem 3.1 follows from the following two propositions.

Proposition 8.1. *There is a $\Lambda \in \mathbf{R}$ such that if $f \in L^2(\mathbf{R}_+^n)$ and $\operatorname{Re} \lambda > \Lambda$ then a weak solution $u \in L^2(\mathbf{R}_+^n)$ to (BVP) is unique.*

Proposition 8.2. *For $q \in \mathbf{Z}_+$ and $\sigma \geq 0$ there is a $\Lambda(q, \sigma) \in \mathbf{R}$ such that if $f \in m^\sigma H^q(\mathbf{R}_+^n; \partial \mathbf{R}_+^n, \gamma)$ and $\operatorname{Re} \lambda > \Lambda(\sigma)$ and if $u \in L^2(\mathbf{R}_+^n)$ with $\operatorname{supp} u \subset \{x_1 \geq 0, |x| < 1\}$ is a weak solution to (BVP) then it follows that $u \in m^\sigma H^q(\mathbf{R}_+^n; \partial \mathbf{R}_+^n, \gamma)$ and the following estimate holds:*

$$\|m^{-\sigma} u\|_{H^q(\mathbf{R}_+^n; \partial \mathbf{R}_+^n, \gamma)} \leq C \|m^{-\sigma} f\|_{H^q(\mathbf{R}_+^n; \partial \mathbf{R}_+^n, \gamma)}$$

where $C = C(q, \sigma, \lambda) > 0$ is independent of f and u .

Proposition 8.1 is an immediate consequence of Lemma 8.3 below. Proposition 8.2 will be proved in the following section.

Lemma 8.3. *Let $u \in L^2(\mathbf{R}_+^n)$ be a weak solution to (BVP) with $f \in L^2(\mathbf{R}_+^n)$. Then we can choose a $\{u_n\} \subset \mathcal{B}^1(\overline{\mathbf{R}_+^n})$ with $u_n \in M$ at $\partial \mathbf{R}_+^n$ so that*

$$u_n \rightarrow u, \quad (L + \lambda)u_n \rightarrow f \quad \text{in } L^2(\mathbf{R}_+^n) \quad \text{as } n \rightarrow \infty.$$

Proof. Let us take a $\chi \in C_0^\infty(\mathbf{R})$ such that $\chi \equiv 1$ near 0 and set

$$u_k = (1 - \chi(km))u, \quad f_k = (L + \lambda)u_k = (1 - \chi(km))f - \tilde{\chi}(km)m^{-1}A_m u$$

where $\tilde{\chi}(t) = t\chi'(t)$. Then u_k is also a weak solution to (BVP) with the right-hand side f_k . Moreover recalling (3.1) and (3.3) we can write

$$(8.1) \quad A_1(x) = x_1 A^{11}(x) + x_2 A^{12}(x), \quad A_2(x) = x_1 A^{21}(x) + x_2 A^{22}(x)$$

where $A^{ij}(x) \in \mathcal{B}^\infty(\overline{\mathbf{R}_+^n})$. Thus it follows from $|m^{-1}A_m| \leq c$ that

$$u_k \rightarrow u, \quad f_k \rightarrow f \quad \text{in } L^2(\mathbf{R}_+^n) \quad \text{as } k \rightarrow \infty.$$

Therefore we may assume that $\operatorname{supp} u \cap \gamma = \emptyset$. Noticing that $\operatorname{rank} A_b(x')$ is constant for $(0, x') \in \partial \mathbf{R}_+^n \cap \operatorname{supp} u$ and using the same arguments as in [11, Theorem 4], we conclude the proof of Lemma 8.3. \square

Theorem 3.2 follows from Theorem 3.1 and the following lemma which is easily checked.

Lemma 8.4. Let $v \in H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)$ and $g = (L + \sigma m^{-1}A_m + \lambda)v \in H^q(\mathbf{R}_+^n; \gamma)$. Then it follows that $v \in H^q(\mathbf{R}_+^n; \gamma)$ and

$$(8.2) \quad \|v\|_{H^q(\mathbf{R}_+^n; \gamma)} \leq C\{\|v\|_{H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)} + \|g\|_{H^q(\mathbf{R}_+^n; \gamma)}\}$$

where $C = C(q, \sigma, \lambda) > 0$.

9. Proof of Proposition 8.2

For the proof of Proposition 8.2, we introduce the following boundary value problem:

$$(BVP)_\sigma \quad \begin{cases} (L + \sigma m^{-1}A_m + \lambda)v = g & \text{in } \mathbf{R}_+^n \\ v(x) \in M(x) & \text{at } \partial\mathbf{R}_+^n \end{cases}$$

Furthermore, in order to get regularity results, we define the following function spaces.

Let $w(x)$ be a function defined in \mathbf{R}_+^n . We introduce the polar coordinates with respect to x_1 and x_2 given by

$$y_1 = \tan^{-1}(x_2/x_1), \quad y_2 = (x_1^2 + x_2^2)^{1/2}, \quad y_j = x_j \quad (j = 3, \dots, n)$$

where $y_1 \in I = (-\pi/2, \pi/2)$. We denote this change of variables by $y = \phi(x)$ and write $\tilde{w}(y) = (w \circ \phi^{-1})(y)$. Note that $\tilde{w}(y)$ is defined in $\mathcal{R}_+ = I \times \mathbf{R}_+ \times \mathbf{R}^{n-2}$. Moreover let us define $\tilde{w}^0(y)$ in $\mathcal{R} = I \times \mathbf{R}^{n-1}$ as

$$\tilde{w}^0(y) = \begin{cases} \tilde{w}(y) & \text{in } \mathcal{R}_+ \\ 0 & \text{elsewhere.} \end{cases}$$

Using this notation we define

$$Y^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma) = \{w \in \mathcal{D}'(\mathbf{R}_+^n); \tilde{w}^0 \in H^q(\mathcal{R}; \partial\mathcal{R})\} \quad (q \in \mathbf{Z}_+)$$

where $H^q(\mathcal{R}; \partial\mathcal{R})$ is the conormal Sobolev space of order q with respect to $\partial\mathcal{R}$. This allows us to norm $Y^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma)$ as

$$\|w\|_{Y^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma)} = \|\tilde{w}^0\|_{H^q(\mathcal{R}; \partial\mathcal{R})}.$$

We shall prove Proposition 8.2 admitting the following three propositions.

Proposition 9.1. For $\sigma \in \mathbf{R}$ there is a $\Lambda(\sigma) \in \mathbf{R}$ such that if $g \in L^2(\mathbf{R}_+^n)$ and $\text{Re}\lambda > \Lambda(\sigma)$ then there exists a weak solution $v \in L^2(\mathbf{R}_+^n)$ to $(BVP)_\sigma$ satisfying

$$(9.1) \quad (\text{Re}\lambda - \Lambda(\sigma))\|v\|_{L^2(\mathbf{R}_+^n)}^2 \leq c\|g\|_{L^2(\mathbf{R}_+^n)}^2$$

where $c > 0$ is independent of σ , λ , g and v .

Proposition 9.2. For $q \in \mathbf{Z}_+$ and $\sigma \geq 0$ there is a $\Lambda(q, \sigma) \in \mathbf{R}$ such that if $g \in Y^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma)$ and $\text{Re}\lambda > \Lambda(q, \sigma)$ and if $v \in L^2(\mathbf{R}_+^n)$ with $\text{supp } v \subset \{x_1 \geq 0, |x| < 1\}$ is a weak solution to $(BVP)_\sigma$ then it follows that $v \in Y^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma)$.

Proposition 9.3. For $q \in \mathbf{Z}_+$ and $\sigma \in \mathbf{R}$ there is a $\Lambda(q, \sigma) \in \mathbf{R}$ such that if $g \in H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)$ and $\text{Re}\lambda > \Lambda(q, \sigma)$ and if $v \in H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)$ is a weak solution to $(BVP)_\sigma$ then it follows that

$$\|v\|_{H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)} \leq C\|(L + \sigma m^{-1}A_m + \lambda)v\|_{H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)}$$

where $C = C(q, \sigma, \lambda) > 0$.

Proof of Proposition 8.2. We first suppose that $f \in C_0^\infty(\mathbf{R}_+^n)$ and that $u \in L^2(\mathbf{R}_+^n)$ with $\text{supp } u \subset \{x_1 \geq 0, |x| < 1\}$ is a weak solution to (BVP). Let us set $g = m^{-\sigma}f$. Applying Proposition 9.1 we can find a weak solution $v \in L^2(\mathbf{R}_+^n)$ to $(\text{BVP})_\sigma$. Then $m^\sigma v$ is also a weak solution to (BVP). Therefore Proposition 8.1 implies that $u = m^\sigma v$, and hence $\text{supp } v \subset \{x_1 \geq 0, |x| < 1\}$. Since it follows from Proposition 9.2 that $v \in Y^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma)$, we have $v \in H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)$. Thus Proposition 9.3 implies that

$$\|v\|_{H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)} \leq C\|g\|_{H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)}.$$

This complete the proof. Next let $f \in H^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n, \gamma)$. By standard limiting arguments we can prove the assertion. \square

Proposition 9.1 is easily checked. The proof of Proposition 9.2 will be given in the following section. Proposition 9.3 follows from the standard a priori estimate (see [9, Section 12]).

The following two lemmas will be used later.

Lemma 9.4. *For $\sigma \in \mathbf{R}$ and $\tau \geq 0$ there is a $\Lambda(\sigma, \tau) \in \mathbf{R}$ such that if $g \in m^\tau L^2(\mathbf{R}_+^n)$ and $\text{Re } \lambda > \Lambda(\sigma, \tau)$ then there exists a weak solution $v \in m^\tau L^2(\mathbf{R}_+^n)$ to $(\text{BVP})_\sigma$ satisfying*

$$(\text{Re } \lambda - \Lambda(\sigma, \tau))\|m^{-\tau}v\|_{L^2(\mathbf{R}_+^n)}^2 \leq c\|m^{-\tau}g\|_{L^2(\mathbf{R}_+^n)}^2$$

where $c > 0$ is independent of σ, τ, λ, g and v .

Lemma 9.5. *For $\sigma \in \mathbf{R}$ there is a $\Lambda(\sigma) \in \mathbf{R}$ such that if $g \in L^2(\mathbf{R}_+^n)$ and $\text{Re } \lambda > \Lambda(\sigma)$ then a weak solution $v \in L^2(\mathbf{R}_+^n)$ to $(\text{BVP})_\sigma$ is unique.*

Lemma 9.4 follows from Lemma 9.6 below. Lemma 9.5 is proved by the same arguments as in the proof of Proposition 8.1.

Lemma 9.6. *For $\sigma \geq 0$ there is a $\Lambda(\sigma) \in \mathbf{R}$ such that if $f \in m^\sigma L^2(\mathbf{R}_+^n)$ and $\text{Re } \lambda > \Lambda(\sigma)$ then there exists a weak solution $u \in m^\sigma L^2(\mathbf{R}_+^n)$ to (BVP) satisfying*

$$(\text{Re } \lambda - \Lambda(\sigma))\|m^{-\sigma}u\|_{L^2(\mathbf{R}_+^n)}^2 \leq c\|m^{-\sigma}f\|_{L^2(\mathbf{R}_+^n)}^2$$

where $c > 0$ is independent of σ, λ, f and u .

Proof. Let us set $g = m^{-\sigma}f$. Applying Proposition 9.1, we can find a weak solution $v \in L^2(\mathbf{R}_+^n)$ to $(\text{BVP})_\sigma$ satisfying (9.1). Then $u = m^\sigma v$ is a desired weak solution to (BVP). \square

10. Proof of Proposition 9.2

In order to prove Proposition 9.2, we introduce the following norm which is equivalent to $\|\cdot\|_{Y^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma)}$: For $0 < \delta \leq 1$ we set

$$\|w\|_{Y^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma), \delta} = \|\tilde{w}^0\|_{\mathcal{R}, q, \tan, \delta}$$

where $\|\cdot\|_{\mathcal{R}, q, \tan, \delta}$ are as in [7, Section 3].

Proposition 9.2 is an immediate consequence of Proposition 10.1 below.

Proposition 10.1. For $q \in \mathbf{Z}_+$, $q \geq 1$ and $\sigma \geq 0$ there is a $\Lambda(q, \sigma) \in \mathbf{R}$ having the following properties: If $g \in Y^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma)$ and $\operatorname{Re} \lambda > \Lambda(q, \sigma)$ and if $v \in Y^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma)$ with $\operatorname{supp} v \subset \{x_1 \geq 0, |x| < 1\}$ is a weak solution to $(\text{BVP})_\sigma$ then the estimate

$$\begin{aligned} & (\operatorname{Re} \lambda - \Lambda(q, \sigma)) \|v\|_{Y^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma), \delta}^2 \\ & \leq c_0 \{ \|g\|_{Y^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma), \delta}^2 + \|v\|_{Y^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma), \delta}^2 \} \\ & \quad + C_1 \{ \|g\|_{Y^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma)}^2 + \|v\|_{Y^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma)}^2 \} \end{aligned}$$

holds for $0 < \delta \leq 1$ where $c_0 = c_0(q, \sigma) > 0$ and $C_1 = C_1(q, \sigma, \lambda) > 0$.

Admitting Proposition 10.1 we give the proof of Proposition 9.2.

Proof of Proposition 9.2. We proceed by induction on q . From Lemmas 9.4 and 9.5 the case $q = 0$ is trivial. Inductively assume the statement is true up to $q-1$. Proposition 10.1 gives $\|v\|_{Y^{q-1}(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma), \delta}^2 \leq C$ with some $C > 0$, and hence we have $v \in Y^q(\mathbf{R}_+^n; \partial\mathbf{R}_+^n \setminus \gamma)$ (see also [7, Section 3]). This proves the assertion for q . \square

Proof of Proposition 10.1. Noticing (3.2) and (3.4) and performing a change of dependent variables we may assume that

$$\begin{aligned} A_b(x') &= \begin{pmatrix} 0 & 0 \\ 0 & x_2 A(x') \end{pmatrix} \quad \text{with some non-singular } A(x'), \\ M_\pm(x') &= M_\pm \quad \text{on } \Gamma_\pm \end{aligned}$$

where M_\pm is a constant linear subspace of \mathbf{C}^N which is independent of x .

Now by the change of variables $y = \phi(x)$, it follows that U , \mathbf{R}_+^n and Γ_\pm are transformed into

$$\tilde{U} = I \times (0, 1) \times \{|y''| < 1\}, \quad \tilde{\mathbf{R}}_+^n = \mathcal{R}_+ \quad \text{and} \quad \tilde{\Gamma}_\pm = \{\pm\pi/2\} \times \mathbf{R}_+ \times \mathbf{R}^{n-2}$$

respectively. Moreover L is transformed into $\tilde{L} = \sum_{j=1}^n \mathcal{A}_j(y) \partial_{y_j} + \tilde{B}(y)$ where

$$\begin{aligned} \mathcal{A}_1(y) &= \sin y_1 \cos y_1 (-\tilde{A}^{11}(y) + \tilde{A}^{22}(y)) - \sin^2 y_1 \tilde{A}^{12}(y) + \cos^2 y_1 \tilde{A}^{21}(y), \\ \mathcal{A}_2(y) &= y_2 \{ \cos^2 y_1 \tilde{A}^{11}(y) + \sin^2 y_1 \tilde{A}^{22}(y) + \sin y_1 \cos y_1 (\tilde{A}^{12}(y) + \tilde{A}^{21}(y)) \}, \\ \mathcal{A}_j(y) &= \tilde{A}_j(y) \quad (j = 3, \dots, n). \end{aligned}$$

Note that if we set $\mathcal{B}(y) = (m^{-1} A_m) \circ \phi^{-1}(y)$ then it follows that $\mathcal{B}(y) = y_2^{-1} \mathcal{A}_2(y)$, and hence $\mathcal{B}(y) \in C^\infty(\overline{\mathcal{R}_+})$. Thus the boundary value problem $(\text{BVP})_\sigma$ is transformed into

$$(\text{BVP})_\sigma \quad \begin{cases} (\tilde{L} + \sigma \mathcal{B} + \lambda) \tilde{v} = \tilde{g} & \text{in } \mathcal{R}_+ \\ \tilde{v} \in M_\pm & \text{at } \tilde{\Gamma}_\pm. \end{cases}$$

The boundary matrix $\mathcal{A}_b(y)$ is given by

$$\mathcal{A}_b(y) = \begin{cases} \pm \mathcal{A}_1(y) &= \begin{pmatrix} 0 & 0 \\ 0 & \pm A(y') \end{pmatrix} & \text{if } y \in \tilde{\Gamma}_\pm \\ -\mathcal{A}_2(y) &= 0 & \text{if } y \in I \times \{0\} \times \mathbf{R}^{n-2}. \end{cases}$$

Therefore the boundary condition of $(\text{BVP})_\sigma$ is maximal positive. Furthermore $\tilde{v}, \tilde{g} \in L^2(\mathcal{R}_+)$ and \tilde{v} is a weak solution to $(\text{BVP})_\sigma$.

Let us extend the boundary value problem $(BVP)_\sigma$ as follows: We still denote by $\tilde{\Gamma}_\pm$ the set $\{\pm\pi/2\} \times \mathbf{R}^{n-1}$. Since $\tilde{a}(y) = a(y_2 \cos y_1, y_2 \sin y_1, y'')$ so we may assume that $\mathcal{A}_j, \tilde{B}, \mathcal{B}, \tilde{H} \in C^\infty(\bar{\mathcal{R}})$. Then the new boundary matrix $\mathcal{A}_b(y)$ is given by

$$\mathcal{A}_b(y) = \begin{pmatrix} 0 & 0 \\ 0 & \pm A(y') \end{pmatrix} \quad \text{if } y \in \tilde{\Gamma}_\pm.$$

Thus we can find a smooth maximal positive boundary space $\tilde{M}_\pm(y)$, $y \in \tilde{\Gamma}_\pm$ such that

$$\tilde{M}_\pm(y) = M_\pm \quad \text{if } y \in \tilde{\Gamma}_\pm, y_2 > 0.$$

Moreover noticing that $\mathcal{A}_2(y) = 0$ on $y_2 = 0$ we have

$$(\tilde{L} + \sigma\mathcal{B} + \lambda)\tilde{v}^0 = \tilde{g}^0 \quad \text{in } \mathcal{R}.$$

Therefore \tilde{v}^0 is a weak solution to the following boundary value problem:

$$(BVP)_\sigma^0 \quad \begin{cases} (\tilde{L} + \sigma\mathcal{B} + \lambda)\tilde{v}^0 = \tilde{g}^0 & \text{in } \mathcal{R}_+ \\ \tilde{v}^0 \in M_\pm & \text{at } \tilde{\Gamma}_\pm. \end{cases}$$

By arguments similar to those in [7] we obtain

Lemma 10.2. *For $q \in \mathbf{Z}_+$, $q \geq 1$ and $\sigma \geq 0$ there is a $\Lambda(q, \sigma) \in \mathbf{R}$ having the following properties: If $\tilde{g}^0 \in H^{q-1}(\mathcal{R}; \partial\mathcal{R})$ and $\operatorname{Re}\lambda > \Lambda(q, \sigma)$ and if $\tilde{v}^0 \in H^{q-1}(\mathcal{R}; \partial\mathcal{R})$ is a weak solution to $(BVP)_\sigma^0$ then the estimate*

$$\begin{aligned} & (\operatorname{Re}\lambda - \Lambda(q, \sigma)) \|\tilde{v}^0\|_{\mathcal{R}, q-1, \tan, \delta}^2 \\ & \leq c_0 \{ \|\tilde{g}^0\|_{\mathcal{R}, q-1, \tan, \delta}^2 + \|\tilde{v}^0\|_{\mathcal{R}, q-1, \tan, \delta}^2 \} + C_1 \{ \|\tilde{g}^0\|_{H^{q-1}(\mathcal{R}; \partial\mathcal{R})}^2 + \|\tilde{v}^0\|_{H^{q-1}(\mathcal{R}; \partial\mathcal{R})}^2 \} \end{aligned}$$

holds for $0 < \delta \leq 1$ where $c_0 = c_0(q, \sigma) > 0$ and $C_1 = C_1((q, \sigma, \lambda) > 0$.

This concludes the proof of Proposition 10.1. □

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